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On stability of the flow around an oscillating sphere

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The stability of the flow resulting from the oscillations of a sphere in a viscous fluid is investigated. The calculation for the transverse oscillations of the sphere is performed in a linear regime and the result in the weakly nonlinear regime is described; the stability in the case of torsional oscillations is considered in the linear regime, where we take torsional oscillations to mean oscillations about a fixed axis through the centre of the sphere. In both cases we assume that the frequency of the oscillations is large, so that the unsteady boundary layer that results is thin. In the transverse case, the linear stability problem depends only on the radial variable and time. Employing Floquet theory we may reduce the system to a coupled infinite system of ordinary differential equations, with homogeneous boundary conditions, the eigenvalues of this system being found numerically. In the torsional case, the linear stability problem again depends only on the radial variable and time, although the angular variation is retained in a parametric form and is determined at higher order. A WKBJ perturbation solution is constructed and the evolution of the amplitude of the vortex is found. The solution is determined by finding a saddle point in the complex plane of the angular coordinate, and thus the critical Taylor number is derived.

1. Introduction

Here we are concerned with the stability of a class of flows which exhibit a phenomenon referred to as steady streaming. In particular, we study the flows induced by the transverse and torsional oscillations of a sphere of radius a, in a viscous fluid of kinematic viscosity ν. These basic flows can be described using a boundary-layer approach similar to that used by Schlichting (1932), as we are considering motion close to a body.

The transverse case follows closely the work of Hall (1984), who considered the transverse oscillations of a cylinder. Hall (1984) was motivated by the experiments of Honji (1981). The main motivation behind the present problem is to obtain further understanding of the instabilities involved in a spherical case. This motion is found to exhibit steady streaming, for further details of which, the reader is referred to Stuart (1966) and Riley (1967). Steady streaming involves the interaction of unsteady components of the disturbance to produce a steady correction to the basic flow in an outer boundary layer. The experiments of Honji (1981) show that the two-dimensional flow induced by the motion of the cylinder is unstable to axially periodic vortices of the Taylor–Görtler type, for sufficiently large frequency oscillations of the cylinder: this is shown theoretically also to extend to the spherical case. These

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instabilities are most likely to occur in a Stokes layer at the position where the surface of the sphere is locally tangential to the motion.

Suppose that a sphere of radius a, oscillates along a diameter, with velocity $U_0 \cos \omega t$. The parameters of importance are β, λ and R_s given by

$$\beta^2 = \frac{2\nu}{\omega a^2}, \quad \lambda = \frac{U_0}{\omega a}, \quad R_s = \frac{U_0^2}{\omega \nu} = 2\lambda^2 \beta^{-2}. \tag{1.1 a–c}$$

The frequency parameter β is taken to be small so that the unsteady boundary layer on the sphere is thin compared to its radius. The parameter λ is the ratio between the amplitude of oscillations and the radius of the sphere, and R_s is the steady-streaming Reynolds number, the importance of which is discussed by Stuart (1966).

Because the resulting layer is thin, the radius of curvature of the path of the particles is the same as that of the sphere, that is $1/a$. Using this assumption we define the Taylor number T by

$$T = \frac{U_0^2}{a \nu^{\frac{1}{2}} \omega^{\frac{3}{2}}} = \frac{R_s \beta}{\sqrt{2}}. \tag{1.2}$$

If the Taylor number is of order one, we may expect to find the instability mechanism described by Taylor (1923) and subsequently by Seminara & Hall (1976); thus we have $R_s = O(\beta^{-1})$, and we focus our attention on the positions where the angular variable $\theta = \frac{1}{2}\pi$, that is where the shear is greatest, and we further confine our attention to the regime $\beta \rightarrow 0$. Making use of (1.1 c) we may infer that $\lambda = O(\beta^{\frac{1}{2}})$. In this limit λ is small, which implies that the layer at the sphere is essentially a Stokes layer. Naturally we employ spherical polar coordinates, where the direction of oscillations is along the line $\theta = 0, \pi$.

From a WKBJ approach we can see that the instability is confined to an $O(\beta^{\frac{1}{4}})$ region. We also include here a brief discussion on nonlinear effects for $R_s - R_{sc} \sim O(\beta^{-\frac{1}{2}})$, where R_{sc} is the critical steady-streaming Reynolds number. Formally, instability occurs when

$$R_s > R_{sc} = R_0 \beta^{-1} + R_1 \beta^{-\frac{1}{2}} + \ldots, \tag{1.3}$$

where R_{sc} is to be determined.

We now consider the torsional case, where the sphere oscillates torsionally around an axis of symmetry with a velocity $U_0 \sin \omega t$. The parameters of importance in this case are the same as those of the transverse case.

The stability properties of a Stokes layer depend on the local geometry. Hall (1978) analysed the stability properties of a flat Stokes layer in the presence and absence of an upper boundary. Tromans (1979) and Cowley (1986) have shown using a quasi-steady approach that Stokes layers are locally unstable to Rayleigh modes. In problems concerning Stokes layers with centrifugal effects, stability analysis employing Floquet theory agrees well with experimental results, as evidenced by comparison of the results of Seminara & Hall (1976) and Honji (1981).

In the torsional case, λ is again taken to be small, thus the layer that occurs at the sphere is again a Stokes layer. In this case the basic flow now has components in two directions, but with no component in the radial direction at leading order. The flow is shown to be locally unstable to instabilities of the Taylor–Görtler type, which are most likely to occur where $\theta = \frac{1}{2}\pi$, that is at the 'equator' of the sphere when the oscillations are about the line $\theta = 0, \pi$.

This work is similar to that undertaken by Papageorgiou (1987), who considered the stability of oscillatory flow in a curved pipe. The instabilities that occur at the outer and inner bend in Papageorgiou's work are similar to those shown to occur here. The perturbation solutions are expanded in terms of β and a WKBJ method can again be employed. This solution is found to become singular in the region of the equator, which suggests the need for an inner expansion. However, certain difficulties are found to arise in connection with this inner solution similar to those found by Soward & Jones (1983) who resolved the problem by the identification of a value of T for which the inner solutions do not exhibit a singularity at the equator.

The rest of this paper is organized as follows: in §2 the transverse case is investigated in the linear regime; in §3 the torsional case is investigated; §4 contains details of the numerical work undertaken in both cases; and finally in §5 the results of the numerical work are presented and some conclusions are given, including a brief discussion of possible extensions to this work.

2. Oscillations in the transverse case; formulation of the linear stability problem in the limit $\beta \to 0$

We employ spherical polar coordinates (r, θ, ϕ) with corresponding velocity components (u, v, w). From physical considerations it may be seen that the basic velocity is independent of the azimuthal coordinate, and the component in this direction is zero. As described in §1 we introduce a Stokes layer on the surface of the sphere of thickness β, and non-dimensionalize the flow quantities in the usual way such that

$$u, v, w \sim U, \quad r \sim a, \quad t \sim 1/\omega, \quad p \sim \rho U^2. \tag{2.1}$$

We may then write the Navier–Stokes equations in the form

$$\frac{\partial u}{\partial t}+\lambda^2\left(u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial \theta}-\frac{v^2}{r}\right)-\frac{w^2}{r} = -\frac{\partial p}{\partial r}+\tfrac{1}{2}\beta^2\left(\nabla^2 u-\frac{2u}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}(v\sin\theta)\right), \tag{2.2a}$$

$$\frac{\partial v}{\partial t}+\lambda^2\left(u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \theta}-\frac{uv}{r}\right)-\frac{w^2}{r}\cot\theta = -\frac{1}{r}\frac{\partial p}{\partial \theta}+\tfrac{1}{2}\beta^2\left(\nabla^2 v+\frac{2}{r^2}\frac{\partial u}{\partial\theta}-\frac{v}{r^2\sin\theta}\right), \tag{2.2b}$$

$$\frac{\partial w}{\partial t}+\lambda^2\left(u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial \theta}+\frac{uw}{r}+\frac{vw}{r}\cot\theta\right) = \tfrac{1}{2}\beta^2\left(\nabla^2 w-\frac{w}{r^2\sin^2\theta}\right), \tag{2.2c}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v\sin\theta) = 0, \tag{2.2d}$$

where we use the parameters introduced in (1.1), and ∇^2 is given by the spherical polar Laplacian with $\partial/\partial\phi \equiv 0$, that is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right).$$

We introduce a boundary-layer variable η defined by $\eta = (r-1)/\beta$. Note from §1 that

$$R_s = \frac{T\surd 2}{\beta}, \tag{2.3}$$

where T is the Taylor number which, as discussed previously, is taken to be of order one, implying that R_s is of order β^{-1}. From the relationships linking β, R_s and λ, we infer that λ is of $O(\beta^{\frac{1}{2}})$. We may use this information to calculate the basic flow quantities, which are expanded in powers of β and are given by

$$v = v_0+\beta v_1+\ldots, \quad u = \beta u_0+\beta^2 u_1+\ldots, \quad p = p_0+\beta p_1+\ldots. \tag{2.4a–c}$$

The leading-order term of the v-component is given by

$$v_0 = \tfrac{1}{2}\sin\theta(\mathrm{e}^{-(1+\mathrm{i})\eta+\mathrm{i}t}-\mathrm{e}^{\mathrm{i}t})+\text{c.c.}, \tag{2.5}$$

where c.c. denotes a complex conjugate. We now perturb the basic flow by adding a perturbation velocity $\boldsymbol{u}'$, which is of the same form as that used in Seminara & Hall (1976) and Hall (1984), and originally by Taylor (1932),

$$\boldsymbol{u}' = \left(\frac{2^{\frac{1}{4}}}{T^{\frac{1}{2}}}\beta^{\frac{1}{2}}\tilde{U}, \tilde{V}, \frac{2^{\frac{1}{4}}}{T^{\frac{1}{2}}}\beta^{\frac{1}{2}}\tilde{W}\right). \tag{2.6}$$

Substituting this perturbation into the Navier–Stokes equations in the Stokes layer, we obtain

$$\mathrm{L}\tilde{U} = \frac{\partial\tilde{P}}{\partial\eta} - 2T^{\frac{1}{2}}v_0\,\tilde{V}\sin\theta + Q_1 + O(\beta^{\frac{1}{2}}), \tag{2.7a}$$

$$\mathrm{L}\tilde{V} = 4\tilde{U}\frac{\partial v_0}{\partial\eta}\sin\theta + Q_2 + O(\beta^{\frac{1}{2}}), \tag{2.7b}$$

$$\mathrm{L}\tilde{W} = \frac{1}{\sin\theta}\frac{\partial\tilde{P}}{\partial\tilde{\phi}} + Q_3 + O(\beta^{\frac{1}{2}}), \tag{2.7c}$$

$$\frac{\partial\tilde{U}}{\partial\eta} + \frac{\beta^{\frac{1}{2}}T^{\frac{1}{2}}}{2^{\frac{1}{4}}}\frac{\partial\tilde{V}}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial\tilde{W}}{\partial\tilde{\phi}} = 0, \tag{2.7d}$$

where L is given by

$$\mathrm{L} = \frac{\partial^2}{\partial\eta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\tilde{\phi}^2} - 2\frac{\partial}{\partial t} - T^{\frac{1}{2}}\frac{\beta^{\frac{1}{2}}}{2}v_0\sin\theta\frac{\partial}{\partial\theta}, \tag{2.8a}$$

and the $\partial/\partial\theta$ terms are neglected in this $\theta = O(1)$ region. The nonlinear terms Q_1, Q_2 and Q_3 are given by

$$Q_1 = 2\tilde{U}\frac{\partial\tilde{U}}{\partial\eta} + \frac{2\tilde{W}}{\sin\theta}\frac{\partial\tilde{U}}{\partial\tilde{\phi}} - \frac{T}{2}\tilde{V}^2, \tag{2.8b}$$

$$Q_2 = 2\tilde{U}\frac{\partial\tilde{V}}{\partial\eta} + \frac{2\tilde{W}}{\sin\theta}\frac{\partial\tilde{V}}{\partial\tilde{\phi}}, \tag{2.8c}$$

$$Q_3 = 2\tilde{U}\frac{\partial\tilde{W}}{\partial\eta} + \frac{2\tilde{W}}{\sin\theta}\frac{\partial\tilde{W}}{\partial\tilde{\phi}}. \tag{2.8d}$$

Here we now include $\tilde{\phi}$ variations so the disturbance is taken to vary on a β-lengthscale, in the azimuthal direction; note that the $O(\beta^{\frac{1}{2}})$ terms included in (2.7), comprise both linear and nonlinear terms. We now linearize (2.7), by neglecting Q_1, Q_2 and Q_3. We also assume periodicity in $\tilde{\phi}$ so that $\tilde{U}$, $\tilde{V}$ and $\tilde{P}$ may be taken to be proportional to $\cos k\tilde{\phi}$, and $\tilde{W}$ proportional to $\sin k\tilde{\phi}$. We combine (2.7*a–d*), eliminating $\tilde{W}$ and $\tilde{P}$, to obtain

$$\frac{1}{\sin\theta}\mathrm{L}\left(\sin\theta\frac{\partial^2\tilde{U}}{\partial\eta^2}\right) - \frac{k^2}{\sin^2\theta}\mathrm{L}\tilde{U} = 2T\frac{k^2}{\sin^2\theta}\tilde{V}v_0\sin\theta - \frac{T^{\frac{1}{2}}}{2^{\frac{1}{4}}}\beta^{\frac{1}{2}}\frac{\partial^2 v_0}{\partial\eta^2}\sin\theta\frac{\partial\tilde{U}}{\partial\theta} + O(\beta^{\frac{1}{2}}), \tag{2.9a}$$

$$\mathrm{L}\tilde{V} = 4\tilde{U}\frac{\partial v_0}{\partial\eta}\sin\theta, \tag{2.9b}$$

where L has $\partial/\partial\tilde{\phi}$ replaced by $\mathrm{i}k$. When compared with the form of L used in Hall (1984) it may be seen that the effective wavelength of the disturbance is $k/\sin\theta$. It can also be seen from (2.9a) that the θ-variation is slow compared with temporal and radial variations. The θ-dependence in $\tilde{U}$ and $\tilde{V}$ may therefore be treated by means of a WKBJ approach, although in this case we are interested in the most unstable mode, and therefore a multiple-scaled technique is employed. We see from (2.9a) that the effective Taylor number for the flow is $T\sin^2\theta$, which has maxima at $\theta = \frac{1}{2}\pi$, and so in the neighbourhood of $\theta = \frac{1}{2}\pi$ the effective Taylor number is given by

$$T(1-(\theta-\tfrac{1}{2}\pi)^2+\ldots). \tag{2.10}$$

Thus $\theta = \frac{1}{2}\pi$ represents a turning point of second order. If the instability is described by the WKBJ method this would imply that there exists a transition layer of thickness $O(\beta^{\frac{1}{4}})$ at this point, this argument being the same as that employed by Hall (1984), and similar to that of Hall (1982).

We now rescale θ, the angular variable, by introducing $\alpha = O(1)$ where $\alpha = \beta^{-\frac{1}{4}}(\theta-\frac{1}{2}\pi)$; v_0 is taken to be evaluated at $\theta = \frac{1}{2}\pi$. We seek a solution of the form

$$\tilde{U} = f_0+\beta^{\frac{1}{4}}f_1+\beta^{\frac{1}{2}}f_2\ldots, \quad \tilde{V} = g_0+\beta^{\frac{1}{4}}g_1+\beta^{\frac{1}{2}}g_2\ldots, \tag{2.11a, b}$$

where now from (2.10) the Taylor number T has the form

$$T = T_0+\beta^{\frac{1}{2}}T_1+\ldots. \tag{2.11c}$$

Substituting (2.11) into (2.9), at leading order we obtain

$$\left(\frac{\partial^2}{\partial\eta^2}-k^2-2\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial\eta^2}-k^2\right)f_0 = 2k^2T_0\,g_0\,v_0, \tag{2.12a}$$

$$\left(\frac{\partial^2}{\partial\eta^2}-k^2-2\frac{\partial}{\partial t}\right)g_0 = 4f_0\frac{\partial v_0}{\partial\eta}, \tag{2.12b}$$

with the boundary conditions

$$f_0 = \frac{\partial f_0}{\partial\eta} = g_0 = 0 \quad \text{at} \quad \eta = 0, \tag{2.12c}$$

$$f_0, g_0 \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty. \tag{2.12d}$$

The eigenvalues of the above system are identical to those of Hall (1984). However, the correction to the wavelength induced by the spherical geometry causes differences in the higher-order equations, because the effective wavelength is

$$k(1+\tfrac{1}{2}\alpha^2\beta^{\frac{1}{2}}+\ldots). \tag{2.13}$$

In order to solve the leading-order eigenvalue problem, we use Floquet theory. The flow quantities are decomposed into modes which are periodic in t; the solutions are taken to be of the form

$$f_0 = A(\alpha)\sum_{n=-\infty}^{n=\infty} f_0^n\,\mathrm{e}^{\mathrm{i}nt}, \qquad g_0 = A(\alpha)\sum_{n=-\infty}^{n=\infty} g_0^n\,\mathrm{e}^{\mathrm{i}nt}. \tag{2.14a, b}$$

Substituting these forms of the solution into the governing equations we obtain an infinite set of coupled ordinary differential equations, given by

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}-k^2-2\mathrm{i}n\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}-k^2\right)f_0^n = T_0\,k^2[g_0^{n-1}(\mathrm{e}^{-(1+\mathrm{i})\eta}-1)]+[g_0^{n+1}(\mathrm{e}^{-(1-\mathrm{i})\eta}-1)], \tag{2.15a}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}-k^2-2\mathrm{i}n\right)g_0^n = -2(1+\mathrm{i})\,\mathrm{e}^{-(1+\mathrm{i})\eta}f_0^{n-1}-2(1-\mathrm{i})\,\mathrm{e}^{-(1-\mathrm{i})\eta}f_0^{n+1}, \tag{2.15b}$$

with relevant homogeneous boundary conditions for $n = 0, \pm 1, \pm 2, \ldots, \pm\infty$. The numerical solution of this system will be discussed in §4.1, and the eigenrelation $k = k(T_0)$ will be computed. Note that $A(\alpha)$, the amplitude, remains undetermined at this order.

At the next order, the partial differential system governing the behaviour of f_1 and g_1 is given by

$$\left(\frac{\partial^2}{\partial\eta^2} - k^2 - 2\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial\eta^2} - k^2\right) f_1 = \frac{T_0^{\frac{1}{2}}}{2^{\frac{1}{4}}}\frac{\partial}{\partial\alpha}\left(v_0\left(\frac{\partial^2}{\partial\eta^2} - k^2\right) - \frac{\partial^2 v_0}{\partial\eta^2}\right) f_0 + 2k^2 T_0 g_1 v_0, \quad (2.16a)$$

$$\left(\frac{\partial^2}{\partial\eta^2} - k^2 - 2\frac{\partial}{\partial t}\right) g_1 = 4f_1\frac{\partial v_0}{\partial\eta} + \frac{T_0^{\frac{1}{2}}}{2} v_0 \frac{\partial g_0}{\partial\alpha}. \quad (2.16b)$$

This system has identical boundary conditions to those of (2.9), but since this system is inhomogeneous we require a solvability condition. It can be seen from the decomposition of f_0 and g_0 into their modes that the system (2.12) decouples into two independent solutions,

$$f_0^n = 0 \quad (n \text{ even}), \quad g_0^n = 0 \quad (n \text{ odd}), \quad (2.17a)$$

$$f_0^n = 0 \quad (n \text{ odd}), \quad g_0^n = 0 \quad (n \text{ even}). \quad (2.17b)$$

The most unstable mode is numerically verified to be (2.17*b*). As in (2.14) we decompose (f_1, g_1) into composite modes, that is

$$f_1 = \sum_{n=-\infty}^{n=\infty} f_1^n(\alpha, \eta)\, e^{int}, \qquad g_1 = \sum_{n=-\infty}^{n=\infty} g_1^n(\alpha, \eta)\, e^{int}, \quad (2.18a, b)$$

implying that only the equations for f_1^n, g_1^{n+1} (n odd) are forced for the system (2.17*b*), that is f_0^n, g_0^{n+1} (n odd) non-zero, and thus $f_1^n = g_1^{n+1} = 0$ for (n odd). The solution for (2.16) can be written as

$$f_1 = \frac{dA}{d\alpha}\hat{f}_1 + B(\alpha) f_0, \qquad g_1 = \frac{dA}{d\alpha}\hat{g}_1 + B(\alpha) g_0, \quad (2.19a, b)$$

where $B(\alpha)$ is a second amplitude function which remains undetermined at this order, but can if necessary be determined at higher order. At the next order we obtain a partial differential system to determine (f_2, g_2), given by

$$\left(\frac{\partial^2}{\partial\eta^2} - k^2 - 2\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial\eta^2} - k^2\right) f_2 = 2k^2 T_0 g_2 v_0 + \frac{T_0^{\frac{1}{2}}}{2}\frac{\partial}{\partial\alpha}\left(v_0\left(\frac{\partial^2}{\partial\eta^2} - k^2\right) f_1 - \frac{\partial^2 v_0}{\partial\eta^2} f_1\right)$$
$$+ 2k^2 v_0\left(T_1 - \frac{\alpha^2 T_0}{2}\right) g_0 + \left(2\frac{\partial^2}{\partial\eta^2} - 2k^2 - 2\frac{\partial}{\partial t}\right) f_0 + 2T_0 k^2 g_0 v_0 \alpha^2, \quad (2.20a)$$

$$\left(\frac{\partial^2}{\partial\eta^2} - k^2 - 2\frac{\partial}{\partial t}\right) g_2 = 4f_2\frac{\partial v_0}{\partial\eta} + \frac{T_0^{\frac{1}{2}}}{2} v_0 \frac{\partial g_1}{\partial\alpha} + k^2\alpha^2 g_0. \quad (2.20b)$$

The last term in (2.20*a*) and the last term in (2.20*b*) do not appear in the equivalent equations in Hall (1984). These terms arise as previously predicted due to the form of the effective wavenumber seen in (2.13). It should also be noted here that there is no term on the right-hand side of (2.20*a*) of the form $v_{0\eta} f_0 \alpha^2$, which occurs in Hall (1984); this is because of cancellation due to the extra terms arising from the effective wavenumber change. The boundary conditions are again those of the system given by (2.12). The forcing terms on the right-hand sides of the equations are synchronous with the solutions of the homogeneous form of (2.20) and a solution for (f_2, g_2) will not in general exist. However, considering the adjoint solution of the first-order system and using the form (2.19) for (f_1, g_1), we obtain the solvability condition

$$\frac{\mathrm{d}^2 A}{\mathrm{d}\alpha^2}+\mu_1\left(T_1-\frac{3\alpha^2}{2}T_0\right)A+\mu_2\alpha^2 A=0. \tag{2.21}$$

This is a version of the equation found in Hall (1984), in which $\mu_2 = 0$. The term μ_1 is given by

$$\mu_1=\frac{\displaystyle\int_{\eta=0}^{\infty}\int_{t=0}^{2\pi} 2k^2 g_0 f_0^{\dagger} v_0\,\mathrm{d}\eta\,\mathrm{d}t}{\displaystyle\frac{T_0^{\frac{1}{2}}}{2^{\frac{1}{4}}}\int_{\eta=0}^{\infty}\int_{t=0}^{2\pi}\hat{f}_0^{\dagger}\left(v_0\left(\frac{\partial^2}{\partial\eta^2}-k^2\right)-\hat{f}_0^{\dagger}\frac{\partial^2 v_0}{\partial\eta^2}\right)\hat{f}_1+\hat{g}_0^{\dagger}v_0\hat{g}_1\,\mathrm{d}\eta\,\mathrm{d}t}, \tag{2.22a}$$

and μ_2 by

$$\mu_2=k^2\frac{\displaystyle\int_{\eta=0}^{\infty}\int_{t=0}^{2\pi}\hat{f}_0^{\dagger}\left(2\frac{\partial^2}{\eta^2}-2k^2-2\frac{\partial}{\partial t}\right)\hat{f}_0+\hat{g}_0\hat{g}_0^{\dagger}\,\mathrm{d}\eta\,\mathrm{d}t}{\displaystyle\frac{T_0^{\frac{1}{2}}}{2^{\frac{1}{4}}}\int_{\eta=0}^{\infty}\int_{t=0}^{2\pi}\hat{f}_0^{\dagger}\left(v_0\left(\frac{\partial^2}{\partial\eta^2}-k^2\right)-\hat{f}_0^{\dagger}\frac{\partial^2 v_0}{\partial\eta^2}\right)\hat{f}_1+\hat{g}_0^{\dagger}v_0\hat{g}_1\,\mathrm{d}\eta\,\mathrm{d}t}. \tag{2.22b}$$

The adjoint functions $\hat{f}_0^{\dagger}$ and $\hat{g}_0^{\dagger}$, satisfy the differential system adjoint to (2.12), that is

$$\left(\frac{\partial^2}{\partial\eta^2}-k^2+2\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial\eta^2}-k^2\right)\hat{f}_0^{\dagger}=4\hat{g}_0^{\dagger}\frac{\partial v_0}{\partial\eta}, \tag{2.23a}$$

$$\left(\frac{\partial^2}{\partial\eta^2}-k^2+2\frac{\partial}{\partial t}\right)\hat{g}_0^{\dagger}=2k^2T_0v_0\hat{f}_0^{\dagger}. \tag{2.23b}$$

The eigenvalues of this system are identical to those of (2.12). The coefficients μ_1 and μ_2 are both functions of k, and hence T_0. Our calculations show that near the minimum value of T_0, μ_1 is positive. We can rearrange (2.21) to give

$$\frac{\mathrm{d}^2 A}{\mathrm{d}\alpha^2}+\mu_1(T_1-\alpha^2 f)A=0, \tag{2.24}$$

where

$$f=\frac{1}{\mu_1}\left(\frac{3\mu_1}{2}T_0-\mu_2\right), \tag{2.25}$$

and so now this equation mimics the corresponding equation in Hall (1984). Equation (2.24) has decaying solutions as $\alpha\to\pm\infty$ if $\mu_1>0$ given by

$$A(\alpha)=A_n(\alpha)=U_n\left(-n-\tfrac{1}{2},2f^{\frac{1}{4}}\alpha\right), \tag{2.26}$$

where U_n is the nth parabolic cylinder function, and the value of T_1 associated with A_n is

$$T_1=T_{1n}=2\frac{n+\frac{1}{2}}{(\mu_1 f)^{\frac{1}{2}}}. \tag{2.27}$$

For the least stable mode, given by $n=0$ we have

$$A_0(\alpha)\sim\exp\left(-\tfrac{1}{2}(\mu_1 f)^{\frac{1}{2}}\alpha^2\right),$$

which exhibits the required exponential decay as $\alpha\to\pm\infty$.

The nonlinear analysis in this case follows that of Hall (1984), and is not included here, but basically we find that the amplitude satisfies the equation

$$\frac{\mathrm{d}^2 A}{\mathrm{d}\alpha^2}+\mu_1\left(\bar{T}-3\frac{\alpha^2}{2}T_0\right)A+\mu_2\alpha^2 A=\gamma A^3, \tag{2.28}$$

where $\bar{T}$ is the value of T_1 predicted by linear theory. This is the same equation as found by Hall (1984), after using the transformation (2.25), to within multiplicative constants of the coefficients. The values of μ_1 and μ_2 are the same as those in the linear amplitude equation (2.22), as would be expected in weakly nonlinear theory. The full analysis involves the extra steady-streaming boundary layer, as studied by Stuart (1966). In this outer layer the perturbation interacts with itself to force a steady component of the basic flow, which can be seen to change the slip velocity at the wall, and thus give rise to premature separation. For details of this phenomenon the reader is referred to Hall (1984).

3. The torsional case

3.1. *The basic flow*

The basic velocity can be taken to have components (u, v, w) which correspond to the spherical polar coordinates (r, θ, ϕ); from physical considerations these velocity components may be assumed to be independent of ϕ. The Navier–Stokes equations are written in terms of spherical polar coordinates, and the sphere is considered to oscillate torsionally about the $\theta = 0, \pi$ axis, thus yielding the boundary conditions

$$u, v = 0; \quad w = U \sin \omega t \sin \theta, \quad r = 1, \tag{3.1 a–c}$$

and

$$u, v, w \to 0, \quad r \to \infty. \tag{3.2 a–c}$$

Balances in the r- and θ-momentum equations yield the following non-dimensionalization in the layer:

$$w \sim U, \quad u, v \sim \frac{U^2}{a\omega}, \quad t \sim \omega^{-1}, \quad r \sim a, \quad p \sim \rho u^2. \tag{3.3}$$

Now dimensionalizing the Navier–Stokes equations using the above scalings we obtain

$$\frac{\partial u}{\partial t} + \lambda^2 \left(u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \theta} - \frac{v^2}{r} \right) - \frac{w^2}{r} = -\frac{\partial p}{\partial r} + \tfrac{1}{2}\beta^2 \left(\nabla^2 u - \frac{2u}{r^2} - \frac{2}{r^2 \sin\theta} \frac{\partial}{\partial \theta} (v \sin\theta) \right), \tag{3.4a}$$

$$\frac{\partial v}{\partial t} + \lambda^2 \left(u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \theta} - \frac{uv}{r} \right) - \frac{w^2}{r} \cot\theta = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \tfrac{1}{2}\beta^2 \left(\nabla^2 v + \frac{2}{r^2} \frac{\partial u}{\partial \theta} - \frac{v}{r^2 \sin\theta} \right), \tag{3.4b}$$

$$\frac{\partial w}{\partial t} + \lambda^2 \left(u \frac{\partial w}{\partial r} + \frac{v}{r} \frac{\partial w}{\partial \theta} + \frac{uw}{r} + \frac{vw}{r} \cot\theta \right) = \tfrac{1}{2}\beta^2 \left(\nabla^2 w - \frac{w}{r^2 \sin^2\theta} \right), \tag{3.4c}$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (v \sin\theta) = 0, \tag{3.4d}$$

where

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right).$$

Again this is a Stokes layer on the surface of the sphere, of thickness $O(\beta)$, and we therefore introduce a scaled variable η, such that $r = 1 + \beta\eta$, where $\eta = O(1)$ and thus in this layer we have

$$u_B = \beta u_0 + \beta^2 u_1 + \ldots, \qquad v_B = v_0 + \beta v_1 + \ldots, \tag{3.5a, b}$$

$$w_B = w_0 + \beta w_1 + \ldots, \qquad p_B = p_0 + \beta p_1 + \ldots. \tag{3.5c, d}$$

Solving for the basic flow quantities at leading order we obtain

$$w_0 = -\tfrac{1}{2} \sin\theta (\mathrm{i}\, \mathrm{e}^{-(1+\mathrm{i})\eta + \mathrm{i}t} + \text{c.c.}), \tag{3.6a}$$

$$v_0 = \cos\theta \sin\theta (\tfrac{1}{4}(1 - \mathrm{e}^{-2\eta}) + \{\mathrm{e}^{2\mathrm{i}t} [\tfrac{1}{8}\mathrm{i} \sqrt{2} (\mathrm{e}^{-\sqrt{2}(1+\mathrm{i})\eta} - 1) + \tfrac{1}{4}\mathrm{i}(1 - \mathrm{e}^{-2(1+\mathrm{i})\eta})] + \text{c.c.}\}). \tag{3.6b}$$

This solution was first discussed by Carrier & DiPrima (1956). Note that these solutions do not satisfy the boundary conditions at infinity; instead a further layer is necessary to satisfy this condition and is of thickness $r-1 = O(\beta^{\frac{1}{2}})$, as shown by Stuart (1966). The basic flow in this outer layer is not required in this linear analysis, although it would be required if the analysis were extended to a weakly nonlinear regime.

3.2. *Formulation of the linear stability problem*

We now perturb the basic flow with a quantity proportional to a small vanishing parameter ϵ_1, so that the resulting analysis is linear. We therefore write

$$\boldsymbol{u} = (u_{\mathrm{B}}, v_{\mathrm{B}}, w_{\mathrm{B}}, p_{\mathrm{B}}) + \epsilon_1(\tilde{u}, \tilde{v}, \tilde{w}, \beta\tilde{p}) + \ldots, \tag{3.7}$$

where now $\lambda^2 = \beta T$ from arguments included in §2. These expansions may be substituted into (3.4) and linearized with respect to ϵ_1. We seek a solution of the form

$$\tilde{u} = b_1(\theta) E\tilde{u}_1 + \beta b_2(\theta) E\tilde{u}_2 + \ldots + \text{c.c.}, \tag{3.8}$$

with similar expressions for $\tilde{v}, \tilde{w}, \tilde{p}$, where the quantities with tildes are functions of η and t only, and E is given by

$$E = \exp\left(\mathrm{i}\beta^{-1}\int_{\theta_0}^{\theta} k(\theta')\,\mathrm{d}\theta'\right), \tag{3.9}$$

where the lower limit in this integral form is arbitrary. The assumed form for E implies that we are considering a disturbance with wavelength of the same order as the thickness of the Stokes layer. We expand the Taylor number as

$$T = T_0 + \beta T_1 + \ldots. \tag{3.10}$$

In this problem we consider disturbances that are independent of ϕ, that is they do not evolve around the sphere. These modes were demonstrated to provide the most unstable linear modes, for example see Seminara & Hall (1976). It is possible that local inflexion instabilities will occur, as discussed by Tromans (1979) and Cowley (1986).

3.3. *The stability problem in the limit $\beta \to 0$*

Substituting the flow form given by (3.7) into (3.4), at leading order we obtain

$$\frac{\partial \tilde{u}_1}{\partial t} + (T_0 v_{\mathrm{B}0}\,\mathrm{i}k + \tfrac{1}{2}k^2)\,\tilde{u}_1 - 2w_{\mathrm{B}0}\,\tilde{w}_1 + \frac{\partial}{\partial \eta}\left(\tilde{p}_1 - \frac{1}{2}\frac{\partial \tilde{u}_1}{\partial \eta}\right) = 0, \tag{3.11a}$$

$$\frac{\partial \tilde{v}_1}{\partial t} + (T_0 v_{\mathrm{B}0}\,\mathrm{i}k + \tfrac{1}{2}k^2)\,\tilde{v}_1 - 2w_{\mathrm{B}0}\,\tilde{w}_1\cot\theta + T_0\frac{\partial v_{\mathrm{B}0}}{\partial \eta}\tilde{u}_1 + \mathrm{i}k\tilde{p}_1 - \frac{1}{2}\frac{\partial^2 \tilde{v}_1}{\partial \eta^2} = 0, \tag{3.11b}$$

$$\frac{\partial \tilde{w}_1}{\partial t} + (T_0 v_{\mathrm{B}0}\,\mathrm{i}k + \tfrac{1}{2}k^2)\,\tilde{w}_1 + T_0\frac{\partial w_{\mathrm{B}0}}{\partial \eta}\tilde{u}_1 - \frac{1}{2}\frac{\partial^2 \tilde{w}_1}{\partial \eta^2} = 0, \tag{3.11c}$$

$$\frac{\partial \tilde{u}_1}{\partial \eta} + \mathrm{i}k\tilde{v}_1 = 0. \tag{3.11d}$$

It is convenient to introduce a vector $\boldsymbol{q}$ to represent the perturbation, in a similar manner to Papageorgiou (1987), and as was originally used by Eagles (1971). We have $\boldsymbol{q} = E(\boldsymbol{q}_1 + \beta\boldsymbol{q}_2 + \beta^2\boldsymbol{q}_3 + \ldots)$, and the $\boldsymbol{q}_i$ are given by

$$\boldsymbol{q}_i = \left[\tilde{p}_i, \frac{\partial \tilde{u}_i}{\partial \eta}, \frac{\partial \tilde{v}_i}{\partial \eta}, \frac{\partial \tilde{w}_i}{\partial \eta}, \tilde{u}_i, \tilde{v}_i, \tilde{w}_i\right]^{\mathrm{T}} \quad (i = 1, 2, \ldots). \tag{3.12}$$

Here and in the rest of the paper superscript T will denote the transpose of a matrix or vector. The first- and second-order linear stability equations may be expressed as

$$\mathbf{J}(\boldsymbol{q}_1) = 0, \tag{3.13a}$$

$$b_2(\theta)\,\mathbf{J}\boldsymbol{q}_2 = \mathbf{L}\boldsymbol{q}_1, \tag{3.13b}$$

where $\mathbf{J}$ is a $(7,7)$ complex matrix differential operator defined by

$$\mathbf{J}\boldsymbol{v} \equiv \boldsymbol{C}\frac{\partial \boldsymbol{v}}{\partial t}+\boldsymbol{A}\boldsymbol{v}+\boldsymbol{B}\frac{\partial \boldsymbol{v}}{\partial \eta}. \tag{3.14}$$

The elements of $\mathbf{L}$ are available from the author, and include second-order basic flow quantities and terms involving $b_1(\theta)$ and $\mathrm{d}b_1/\mathrm{d}\theta$. The matrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are also $(7,7)$ and are given by

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & T_0 v_{\mathrm{B0}}\,\mathrm{i}k+\frac{1}{2}k^2 & 0 & -2w_{\mathrm{B0}} \\ \mathrm{i}k & 0 & 0 & 0 & T_0\dfrac{\partial v_{\mathrm{B0}}}{\partial \eta} & T_0 v_{\mathrm{B0}}\,\mathrm{i}k+\frac{1}{2}k^2 & -2w_{\mathrm{B0}}\cot\theta \\ 0 & 0 & 0 & 0 & T_0\dfrac{\partial w_{\mathrm{B0}}}{\partial \eta} & 0 & T_0 v_{\mathrm{B0}}\,\mathrm{i}k+\frac{1}{2}k^2 \\ 0 & 1 & 0 & 0 & 0 & \mathrm{i}k & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.15}$$

$b_{ij} = 0$ for all $i = 1,2,\ldots,7, j = 1,2,\ldots,7$, except $b_{11} = 1, b_{12} = -\frac{1}{2}, b_{23} = -\frac{1}{2}, b_{34} = -\frac{1}{2}$; $c_{ij} = 0$ for all $i = 1,2,\ldots,7$, $j = 1,2,\ldots,7$, except $c_{15} = 1$, $c_{26} = 1$, $c_{37} = 1$.

The system (3.13*b*) has a solution only if the appropriate adjoint condition is satisfied, from which we can derive a differential equation to determine the behaviour of the leading-order amplitude function b_1. We introduce $\mathbf{J}^{\dagger}$, which is the operator associated with the adjoint system of (3.13*a*) as follows:

$$\mathbf{J}^{\dagger} \equiv -\boldsymbol{C}^{\mathrm{T}}\frac{\partial}{\partial t}+\boldsymbol{A}^{\mathrm{T}}+\boldsymbol{B}^{\mathrm{T}}\frac{\partial}{\partial \eta}. \tag{3.16}$$

If $\boldsymbol{V}$ is the solution of the first-order adjoint system we have

$$\mathbf{J}^{\dagger}(\boldsymbol{V}) = 0, \tag{3.17}$$

and since $\boldsymbol{q}_1$ is a solution of (3.13*a*)

$$\boldsymbol{V}^{\mathrm{T}}\mathbf{J}(\boldsymbol{q}_1) = 0. \tag{3.18}$$

Integrating (3.18), over the entire Stokes layer ($\eta = 0, \infty$) and over one time period ($t = 0, 2\pi$) we can derive a solvability condition. The previously mentioned orthogonality condition for (3.13*b*) leads to the result

$$\int_{\eta=0}^{\infty}\int_{t=0}^{2\pi} \boldsymbol{V}^{\mathrm{T}}\,\mathbf{L}\boldsymbol{q}_1\,\mathrm{d}t\,\mathrm{d}\eta = 0. \tag{3.19a}$$

From this it is possible to derive an ordinary differential equation governing the leading-order amplitude function $b_1(\theta)$, namely

$$f_1(\theta)\frac{\mathrm{d}b_1}{\mathrm{d}\theta}+f_2(\theta)\,b_1 = 0, \tag{3.19b}$$

where f_1, f_2 are double integrals over the entire problem space.

We have a differential equation for the leading-order amplitude, so we can determine the size of the vortex for a critical Taylor number, T_0 (for a fixed θ) given the wavenumber, $k(\theta)$. It is expected that vortices will be of larger amplitude in certain positions (i.e. for certain values of θ). Owing to the form of E, (3.9), and the fact that k may be complex, leads to the possible decay of the vortex amplitude with respect to θ, and it is consequently necessary to solve the problem in the complex θ-plane; for details of this calculation see §4.

If we attempt a similar analysis to that of Papageorgiou (1987), we arrive at the same conclusion. We find an irregularity at the origin in the amplitude equation. Briefly, expand in terms of $\theta_1^{\frac{1}{2}}$, which leads to the amplitude equation

$$(\theta_1^{\frac{1}{2}}+\alpha_1\theta_1+\ldots)\frac{\mathrm{d}b_1}{\mathrm{d}\theta_1}+\tfrac{1}{4}(\theta_1^{-\frac{1}{2}}+\alpha_0+\ldots)\,b_1=0, \tag{3.20}$$

where α_0, α_1 are known constants. A solution of this equation may be shown to behave in the form

$$\Gamma_1\,\theta_1^{-\frac{1}{4}}\exp\left((\alpha_1-\alpha_0)\,\theta_1^{\frac{1}{2}}\right), \tag{3.21}$$

where Γ_1 is a constant. This form of solution shows that as $\theta_1\rightarrow 0$, b_1 becomes singular, suggesting that a different structure occurs close to $\theta_1=0$. To determine this structure it is again necessary to extend the solution into the complex θ-plane. By the identification of a complex value of θ at which a saddle point occurs, it is necessary to treat this problem in a similar manner to Soward & Jones (1983). It can be shown that the amplitude equation becomes,

$$I_M\,d_{1\Theta\Theta}+I_N\,\Theta d_{1\Theta}+I_P\,d_1+I_R\,d_{1\Theta}+I_S\,\Theta d_1+I_Q\,\Theta^2 d_1=0, \tag{3.22}$$

where the I are double integrals over the entire problem space, Θ is an inner real angular variable and d_1 is the first-order perturbation amplitude. We can show that the solution for d_1 is given by

$$d_1(\Theta)=\mathrm{e}^{-\alpha^2\Theta^2}\,\hat{F}(\Theta), \tag{3.23}$$

where α is a real constant and $\hat{F}(\Theta)$ is a Hermite polynomial. The exponential factor in (3.23) ensures that the amplitude decays as $\Theta\rightarrow\pm\infty$. The identification of the saddle point is attempted numerically in §4.2. Note that the point is at an $O(1)$ distance from the real θ-axis; further arguments as to the validity of this solution may be found in Papageorgiou (1987).

4. Numerical solution of the eigenvalue problems

4.1. *Numerics for the transverse case*

The numerical solution of the eigenvalues for this system is tackled in similar manner to the torsional case, and may be written symbolically as

$$F(T_0, k)=0. \tag{4.1}$$

As mentioned previously we calculate the asymptotic form of the solution at some η_∞, taken to be large enough that we may neglect exponentially small terms in the basic flow. This solution is then integrated towards $\eta=0$, using a fourth-order Runge–Kutta scheme, and then discrepancies involved in this calculation are used to

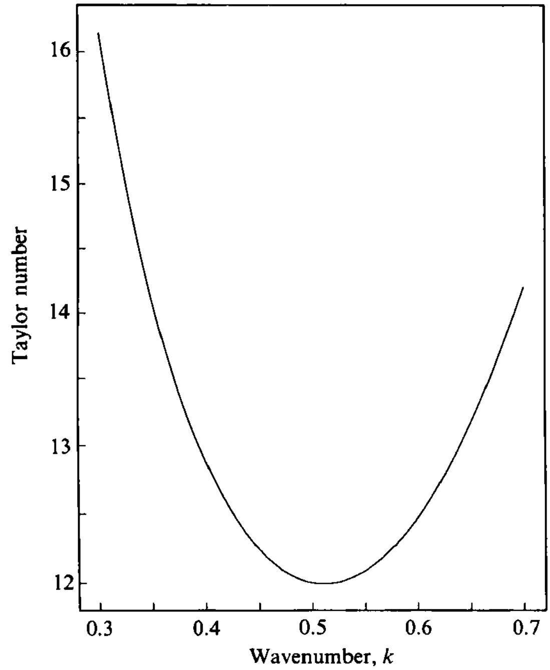

FIGURE 1. Neutral curve for the transverse case, with critical Taylor number 11.99 occurring at wavelength 0.51.

iterate to obtain a better estimate of T_0, for a given k. When exponential terms are removed from (2.12), we are required to solve

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}-k^2-2\mathrm{i}n\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}-k^2\right)f_0^n = -T_0\,k^2(g_0^{n-1}+g_0^{n+1}), \tag{4.2a}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}-k^2-2\mathrm{i}n\right)g_0^n = 0. \tag{4.2b}$$

We only retain a finite number of Fourier modes, and for each one of these we find there are three independent solutions which decay as $\eta \rightarrow \infty$. If $2M+1$ modes are retained we obtain $6M+3$ independent solutions of the system, which can each be integrated to $\eta = 0$ as described previously. We now satisfy the $6M+2$ boundary conditions at the sphere, and the 'extra' solution at $\eta = 0$ will only be satisfied if (4.1) is satisfied, and so this condition may be used to iterate T_0. The secant method was found to be adequate for this procedure.

Notice, as previously stated in (2.17), that the system decouples, and we only investigated the system (2.17*b*), since this produces the most unstable modes. Note that by only considering this 'half' of the system we reduce the order of the system to be solved numerically to $3M+2$. In this calculation we used $M = 6$ and $\eta_\infty = 10$ and forty steps were used in the Runge–Kutta scheme, these values being found to be sufficient by increasing them and checking that consistent results were obtained. The neutral curve is given by figure 1, and its minimum is given by

$$T_{0\mathrm{c}} = 11.99, \quad k_\mathrm{c} = 0.51, \tag{4.3}$$

this result being consistent with that of Hall (1984). Note it was necessary to effect this calculation as there is no transformation between any integral in Hall (1984) and that for μ_2. However, it is not necessary to perform the calculation for the evaluation

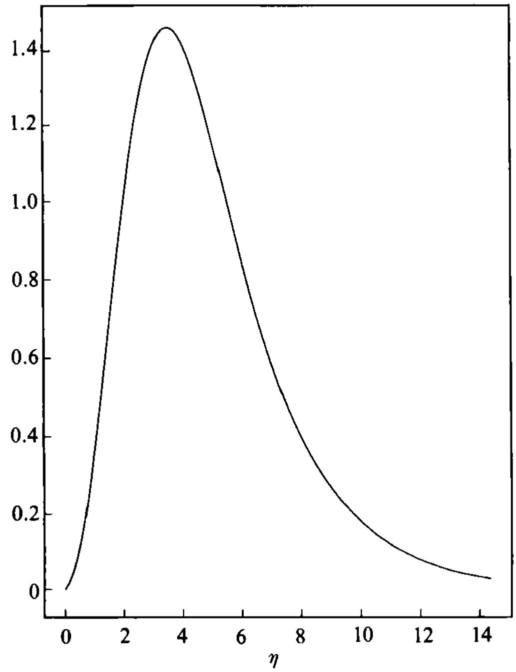

FIGURE 2. First-order eigenfunction f_0^0, normalized so that u'' is unity at the sphere, evaluated at the critical Taylor number.

of μ_1 and γ as by use of the correct size disturbance we can make these expressions linear multiples of the corresponding results of Hall (1984). These constants are given by

$$\mu_1 = 0.0098, \quad \mu_2 = -0.013655561, \quad \gamma = -0.087(2^{\frac{9}{4}}) = -0.4138. \quad (4.4\,a\text{–}c)$$

It is necessary to solve the adjoint system given by (2.23) (which produces the same eigenvalues as the original system and so serves as a check on consistency). We combine the results of (4.4*a*, *b*) to obtain

$$f = 17.998. \quad (4.5)$$

We also include plots of the function f_0^0 (figure 2), and of real and imaginary parts of the first harmonic g_0^1 (figure 3). These functions are normalized so that $\partial^2 f_0^0/\partial\eta^2 = 1$ on $\eta = 0$.

4.2. *The torsional case*

As is to be expected the methods employed in this section follow closely those employed by Papageorgiou (1987), since we are solving a system involving the same mechanism but with a different basic flow. The leading-order eigenvalue problem is solved numerically, subject to the boundary conditions of no slip at the sphere and exponential decay at infinity. The method used is a modification of that described in §4.1, with a modification for different asymptotic behaviour at infinity. It is also necessary to progress in ξ, where ξ is the distance from the real θ-axis, in order to determine the saddle point, although in fact ξ is purely parametric, and so solving the eigenproblem at any fixed ξ introduces no additional difficulties, and the value at a previous ξ can be used to provide the initial estimate for the eigenvalue.

The neutral curve for $\xi = 0$ is given here as figure 4; note that the minimum occurs at $k_c = 1.19$, where the critical Taylor number has the value $T_{0c} = 161.95$. We now note that the minimum value of the Taylor number may not occur at the same value of k for all values of ξ, as indeed was the case in Papageorgiou (1987). It is found

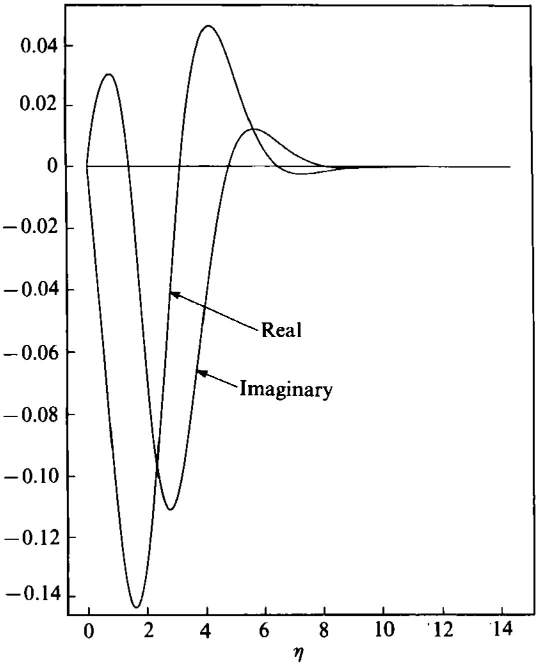

FIGURE 3. Second-order eigenfunction g_0^0, real and imaginary parts, computed at the critical Taylor number.

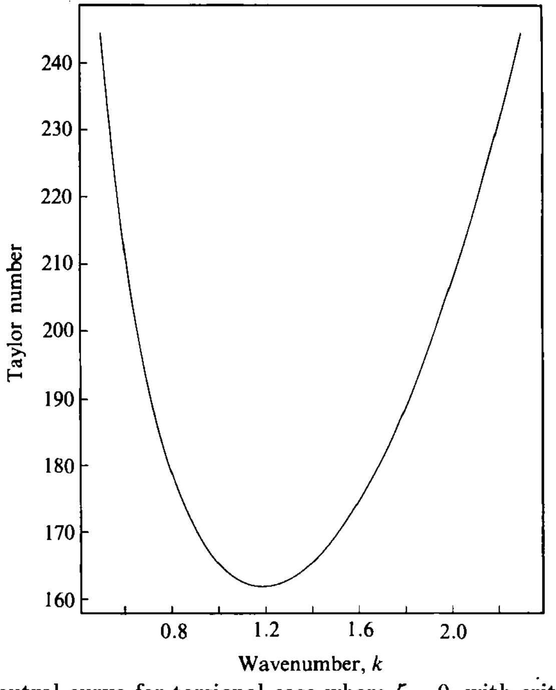

FIGURE 4. Neutral curve for torsional case where $\xi = 0$, with critical Taylor number $T_0 = 161.95$ occurring at $k_c = 1.19$.

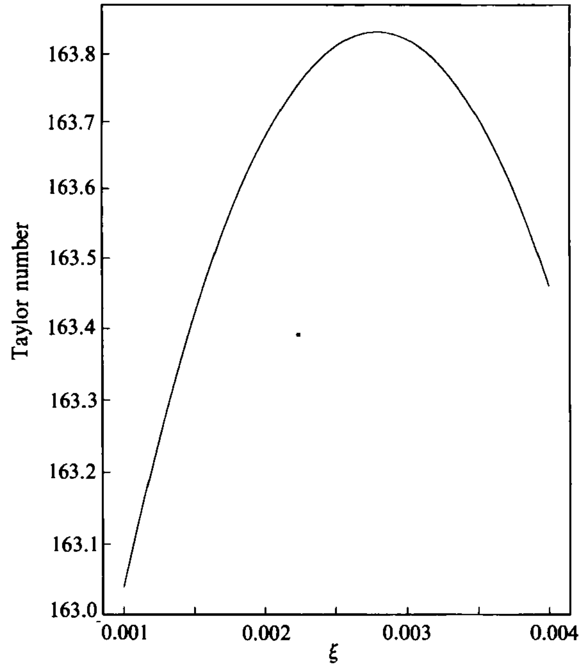

FIGURE 5. Variation of critical Taylor number T_0, with ξ.

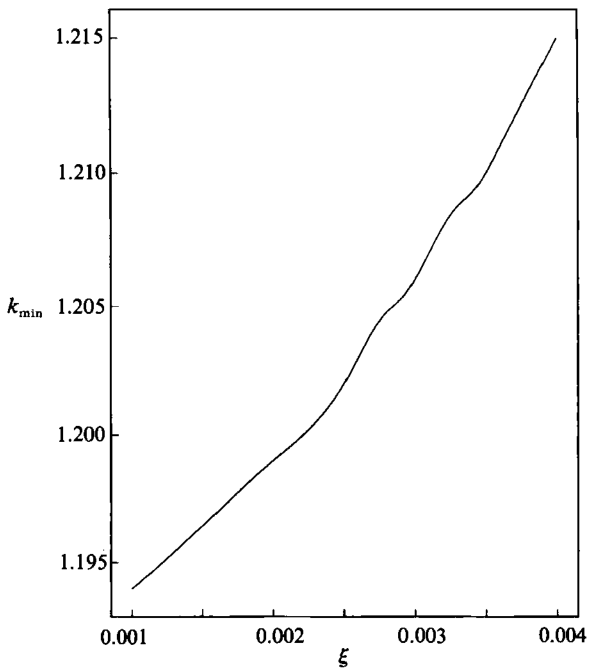

FIGURE 6. Variation of critical wavenumber k, with ξ.

necessary to find the minimum value of the Taylor number for each value of ξ, the result of which is displayed in figure 5. We also show the values of k at which the minimum occurs against ξ in figure 6. The saddle point occurs at,

$$T_{0c} = 163.82, \quad k_c = 1.2048, \quad \xi_c = 0.00284. \tag{4.6}$$

Note that the value of ξ at which the minimum occurs is much smaller than that occurring in Papageorgiou (1987), because of the size of the value of T_0 occurring in the weighting factors associated with the basic flow quantities.

5. Conclusions

We have shown that in both the torsional and transverse problems the oscillatory viscous flow considered, interacting with the rigid boundaries of convex curvature, may become unstable to Taylor–Görtler vortices. In the transverse case we have shown that the flow is linearly unstable to the mechanism described about the 'rim', that is essentially where the slip velocity is at a maximum. Similarly in the torsional case the flow is shown to be linearly unstable to a vortex structure at the 'equator'.

In the transverse case we consider an instability local to $\theta = \frac{1}{2}\pi$ and the vortices are shown to be confined to a layer of thickness $\beta^{\frac{1}{4}}$ surrounding this location. We can show that

$$R_{\mathrm{sc}} = 16.956(\beta^{-1}+O(\beta^{-\frac{1}{2}})) \tag{5.1}$$

and

$$\lambda_{\mathrm{c}} = 2.911(\beta^{\frac{1}{2}}+O(\beta^{\frac{3}{4}})). \tag{5.2}$$

So if R_{s}, the steady-streaming Reynolds number is greater than its critical value R_{sc}, then the vortices grow exponentially in the aforementioned layer. The amplitude of the linear disturbance is given by a parabolic cylinder function and thus the vortex decays away from the region of vortex activity.

In the torsional case the amplitude of the instability, in the linear case, is found to be governed by an exponential factor and an algebraic term varying with an inner angular variable, which ensures decay away from the region of activity, but we also have a singularity in the amplitude as θ_1 (the inner angular variable) tends to zero. It is found to be necessary to extend the analysis into the complex θ-plane, as in Soward & Jones (1983). We can then resolve the problem of the singularity and the amplitude is found to be an exponential factor multiplying a Hermite polynomial, thus we have the required decay at infinity and thus the singularity problem is resolved.

The mechanisms involved here are of a similar type to those in Papageorgiou (1987) and Hall (1984), included in which are comparisons with relevant experiments, to which the reader is referred for further discussion.

The results of the nonlinear calculation given here apply only for the transverse case, and are consistent with those of Hall (1984), which includes detailed discussions of the numerical solution of the weakly nonlinear amplitude equation. It should be noted at this point that this result was not as expected owing to the subcritical nature of the instability.

Both problems are prone to a phenomenon, induced by the oscillatory motion, known as steady streaming. Associated with which is the steady-streaming Reynolds number, R_{s}. The importance of this parameter was first explained by Stuart (1966), who showed that for $R_{\mathrm{s}} \gg 1$ the steady streaming decays to zero in an outer boundary layer of relative thickness $R_{\mathrm{s}}^{\frac{1}{2}}$, the instability described in this paper occurs for $R_{\mathrm{s}} \gg 1$. Stuart (1966) stated that Reynolds stresses and the instability are responsible for driving the steady streaming, as reported in Hall (1984, 1986) which may invoke premature separation of the steady streaming boundary layer.

It may be interesting to consider different geometries, for example extending both problems to ellipsoidal coordinate systems. Certain difficulties are encountered in the initial formulation of these problems, for example the derivation of the basic flow.

Hall (1984) discussed the extension of the transversely oscillating cylinder to that of an ellipsoidal cylinder, although in that case it is possible to transform this cylinder back into one of circular cross-section. This is not possible in these higher dimensional cases, since the basic flow will involve the angular coordinates in other than purely parametric forms. It is most likely that we will be able to extend these problems into geometries with small eccentricities from the spherical case, although it also seems preferable to restrict our attention to solids of rotation, thus reducing the number of degrees of freedom associated with the problem by one.

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